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PRESIDENTIAL ADDRESS.

ON THE GEOMETRICAL METHOD.

As I can pretend to only limited practical experience in the matter of teaching the foundations of Mathematics, and of Geometry in particular, I have thought that the time allowed for my remarks may be advantageously occupied in a rapid survey of what the geometrical method has accomplished, and the estimation in which it has been held at different periods of the history of the development of Mathematical Science. Thus, avoiding expert questions relating to Education, we may begin by considering from a general standpoint what is implied in the science of Geometry, for the right ordering of which, in its educational relations, this Association has been working for a long series of years. It would, I presume, be against the instincts of our members to interpret the subject in any narrow sense. Geometry is usually based on the partly mathematical, partly logical, framework which has descended to us from Greek times, and which is chiefly associated with its earliest great systematization in the work of Euclid. But modern development would perhaps render any definition of its scope incomplete that did not include all methods in formal and physical Mathematics which proceed by direct contemplation of the relations of the subjects considered, instead of through representations of them that are merely quantitative or numerical.

From this standpoint I think it will appear that the geometrical method has always been the one for which British predilection has strongly declared itself. After the early period, when Englishmen like Recorde assisted in forming the foundations of Algebra, it is noticeable how few of the improvers of the machinery of Analysis have been of British race. Even the Infinitesimal Calculus was directly based by its English founder on geometrical ideas, in marked contrast to the analytical notation which was a chief merit of Leibnitz's rival scheme. During the whole of last century we had few cultivators of algebraical Analysis to place opposite the Bernoullis, Euler, Lagrange, and a host of Continental developers of that field. In the present century we have indeed been instructed in, and have taken full advantage of, the new weapons of analysis, but that at a time when they had nearly reached maturity of form,

and opportunities of improving them were not so numerous. The fundamental analytical improvements due to Hamilton had intimate relation to the geometrical standpoint; and there perhaps remains in addition to these only the modern algebra of Cayley and Sylvester, which may be claimed as an analytical product chiefly British.

But alongside this national preference for graphical methods, which is associated no doubt with the practical and concrete character of the British genius, there is the remarkable fact that the narrower science of Pure Geometry proper has hardly formed, at any rate during the middle part of this century, any part of academic instruction in England. To take an illustration, during that time the geometrical portion of the Mathematical Tripos Examination at Cambridge was meagre in the extreme. It consisted of a few propositions from Euclid, to be demonstrated in the conventional form adopted by the English translations, and of a few original geometrical problems —riders as they were called—attached to the propositions, which were often more of the nature of puzzles than illustrations of geometrical ideas—a circumstance due in part to the absence of any organized treatment of geometrical method, and which were usually of excessive difficulty. It was, in fact, in the University of Dublin that the subject of Pure Geometry in the middle of this century showed organic growth, partly arising from the intimate connection maintained with the French school; and it was, no doubt, mainly through the influence of the writings of Dublin Mathematicians that Pure Geometry has regained an adequate place in the English educational system. Reference should, however, not be omitted to the continuous cultivation of Geometry in the Scotch Universities from the times of the Gregories and Maclaurin downwards. There is, perhaps, some ground for an induction that the Celtic mind is specially apt at the subtle kind of ingenuity which is necessary to the successful cultivation and explanation of the ramifications of this science; while the Teutonic mind is, with some exceptions, more attracted to searching after the ultimate philosophical basis of geometrical cognition than to exploring the immense landscape that is dominated by the axioms and postulates of Pure Geometry.

I was about to express an opinion that Geometry is the queen of the Sciences; that dignity is, however, already bestowed. We may, at any rate, recall the recognition which has been granted in all ages to the claim that Geometry sets the example and points the method which every science involving long trains of deductive reasoning should follow. During the time of early development of a branch of knowledge, shorter and more empirical methods of explanation will prove economical; but the

perfect form of the completed structure will usually approximate more and more to the geometrical ideal. According to Plato himself, the secrets of Philosophy are closed to those who do not bring with them a geometrical training. And in the modern age, when a Descartes or a Spinoza wishes to subject his speculation to the most searching scrutiny, or to claim for it irrefragable demonstration, he by instinct expresses it in severely geometrical form.

When it is an abstract survey of the chain of connection between thought and language that is aimed at, we are led to the study of the science of Logic and to a formulation of the principles of logical Analysis. But when it is the orderly development of a complicated system of reasoning that has to be undertaken, the desirable auxiliary is rather a perception of the actual scientific method which has served to express and to guide the growth of the most far-reaching and widely ramified train of pure deduction that the human mind has accomplished.

It is, however, not difficult to understand the enthusiasm with which, three quarters of a century ago, the group of younger men, among whom were prominent Woodhouse, Herschel, Peacock, and Babbage, threw themselves into the task of introducing the Continental Analysis into the course of study at the Universities of this country. The range of subjects to which the Newtonian geometrical methods were at that time applied in the schools, was for the most part limited to portions of the *Principia* itself, and to those portions in which the genius of its author had left least to be gleaned, except in matters of detail, by his followers. The problems then demanding solution, in Celestial Mechanics, were of unrivalled precision; but numerical calculation by unaided geometrical methods had been pushed about as far by Newton himself as it was possible for human ingenuity to carry it. What was chiefly wanted was the development of a new Calculus, which could compute the arithmetical results that flow from the theory of Universal Gravitation, and find out whether they corresponded, or were in discord, with the precise facts of Astronomy. It was, in fact, necessary that the direct geometrical exposition of Newton should give place to a method in which abstraction was made of the actual phenomena in order that the mind might be concentrated solely on the process of calculation. The Analysis of the Continental mathematicians was thus historically, to a great extent, a product of the necessities of this problem of Physical Astronomy. It became a tremendous engine for computation; and its very success for that purpose often somewhat obscured the deeper scientific aspects of its methods, and so left a rich harvest to be gleaned from the writings of the analysts by subsequent investigators, whose more geometrical

method of training incited them to penetrate further beneath the symbolism.

The progress attained by this analytical Calculus must have seemed almost miraculous to minds in this country that were familiar only with the purely geometrical methods, and had grown used to the limitations of their power in the direction in which the efforts of theoretical astronomers then lay. But the notion sometimes expressed that the Eighteenth Century was one of mathematical stagnation in Great Britain is one for which there is perhaps not much warrant, at any rate, on the Physical side. The foundations of the science of Electricity were, at that very time, being steadily and quietly laid by Cavendish, and those of the science of Heat by Black and others. And when at the very beginning of the present century theoretical Optics awoke to new life, which heralded the introduction of the modern Molecular Physics, the first long steps were taken by a Newtonian, whose erudition in the Continental Analysis was only less wonderful than the strength of his preference for the more direct geometrical methods of the Newtonian school. And when, again, the science of Electricity, and with it the whole method of modern Mathematical Physics, had to be reconstructed in the presence of a now wide and keenly interested scientific opinion, the place of honour came to the lot of an investigator whose mathematical equipment was confined to a keen perception of space-relations, who had the advantage of knowing nothing of the methods of algebraical Analysis, as at that time developed, and so was not biassed by the spell of their power, which, however effective in Astronomy, tended to lead into a wrong track in the newer subjects. The discovery of new principles in Natural Philosophy by the purely abstract process of algebraical reasoning require indeed the highest qualities of intellect, and has perhaps been achieved to any considerable extent only by the great masters in Science. In the case of ordinary minds the study of Algebra usually only leads to a capacity to follow their reasoning and apply it to particular instances. In the geometrical method it is the thing itself that is before the mind instead of a numerical symbol of the thing; the training is in the direct survey of the relations and connections of different things, and in the simplest expression of these relations, as witness the subjects of Cartography and Graphical Mechanics. It has even happened that many of the most fertile ideas in modern algebraic Analysis have been directly transplanted from descriptive physical theories.

It would seem that the line of development of abstract mathematical thought in its higher branches has at length begun to turn aside from the purely analytical operations of differentiation and integration into which it was guided by the calculations of

Physical Astronomy. The present aim is rather to establish a broader basis for the results of Analysis, and to reach wider points of view, by taking a survey of the distribution of the quantities it deals with, by the aid derived from representing them as mapped out in space, and thus exploring their relations through the geographical connections of the regions with which they become associated. It is perhaps safe to assert that the fabric of higher Pure Mathematics could hardly have its present vital qualities of growth apart from the help it receives from geometrical intuition. It has happened that when the more artificial analytical methods were in danger of sinking under the weight of their accumulated results, the powerful resources of space-perception have come to the rescue, and started a new line of progress.

Each successive development of symbolical calculation seems to be limited in scope, and finally to reach a stage in which the mind cannot follow it further until it is revivified by being brought back to the fountain of direct intuition. The artificial ideas of the Calculus again give place to contemplation of the actual aspects of phenomena in time and space, but in the light of increased knowledge; and when a right orientation has thus been secured, a new harvest of numerical or quantitative results is once more within the power of Analysis. The algebraical method in this sense never supersedes the geometrical. The more vigorous and fully developed our geometrical ideas become, the more effectively shall we be able to co-ordinate the phenomena of experience, and thereby gain fruitful ground for the operations of the Calculus. But to be efficient for this purpose the geometrical ideal must be kept pure, and in constant relation to intuitive perception: a highly specialized geometrical theory is as feeble an instrument for tracing the wide general relations of things as abstract Algebra itself. The recognition of this informing power of Geometry as distinguished from the numerical or computing power of Analysis has in recent years received wide exemplification in the domain of the engineer. Owing to the somewhat vague and roughly approximate nature of the data usually at his disposal, the mechanical engineer very rarely wants exact numerical solutions of his problems; and the general views that can be derived from the simple inspection of geometrical constructions very often carry with them information which is sufficient for his purposes, in cases where a numerical solution would be far beyond the range of analytical calculation.

The broad features of the distinction between the geometrical and the analytical methods of mathematical reasoning reveal themselves strikingly in the two main types of treatises on general Physics. In this country the model, as illustrated by such books, of different epochs, as Young's *Lectures on Natural Philosophy*,

and Thomson and Tait's *Natural Philosophy*, has usually been a description and correlation of facts and laws over a wide field, enforced and illustrated by the application of such analytical methods as are capable of being presented in compact and handy form, and are therefore presumably within the reach of most people who are qualified to cultivate the subject. There is another type of treatise, intended only for the small body of competent specialists, which first surveys the limited field that it is to cover, then assigns a symbol to represent each variable in the problem, forms its equations and sails away into the ocean of algebraic Analysis, ultimately attaining results perhaps few, but usually important, which for a long period may have to be taken on faith by the great majority of students. Even here, however, the algebraic method is not so self-centred as might appear. As already mentioned, its processes were originally invented mainly in order to deal with the problem of the inequalities in the movements of the bodies constituting the Solar System. A very precise descriptive knowledge of these inequalities was already in existence, as the result of centuries of astronomical observation: and the formal or geometric aspect of the phenomena, as thus crystallized by observation and reflection, was the best possible guide to the character of the Analysis, which would be most appropriate to explain and verify them. And afterwards it was not so difficult to pass on from the extremely simple and precise relations of the Sun and planets, to the generalized conditions of any dynamical system; and with the help of the geometrical principle of Virtual Work, and the semi-geometrical principle of Least Action, which had long formed part of the philosophy of descriptive Mechanics, to finally arrive at the great generalizations of Lagrange and Hamilton in Analytical Dynamics, whose application and verification throughout the range of physical phenomena is now going on.

This view of the growth of Analytical Dynamics is in striking contrast to Lagrange's final presentation of its results in the *Mécanique Analytique.* A cardinal announcement in his Preface is the famous and occasionally much censured sentence, *On ne trouvera point de Figures dans cet ouvrage.* The framework of Dynamics has been cast by him into such a form, that when once the co-ordinates sufficing to express the positions and properties of the bodies composing the dynamical system are given, its subsequent history is deduced by a regular analytical process. "Les méthodes que j'y expose ne demandent ni constructions, ni raisonnements géométriques ou mécaniques, mais seulement des opérations algébriques, assujéties à une marche regulière et uniforme." This sentence represents the crowning success of an analytical method, to reduce everything to a calculus in which no further examination or independent

consideration of the data is required. And here it is to be observed, on the other hand, that such an analytical generalization may lead into new regions that might not have been discovered at all by methods which kept closer to observation. The co-ordinates of a dynamical system may have their meaning widely generalized; and a comparison of the results of such a process with the facts of Nature may lead to yet more sweeping generalizations, and thus gather into the domain of pure Dynamics phenomena whose dynamical aspect would hardly have been unravelled by direct methods. It is in this kind of way that a dynamical basis is gradually being evolved for the phenomena of Heat and Electricity and Chemical Action, and will possibly some day be worked out for the secrets of the constitution of Matter itself.

A striking instance of the generalizing aspect of Analysis is in fact furnished by Lagrange's own treatment of the subject of Kinematics, or the pure geometrical theory of Motion. As has been already mentioned, he discards altogether the use of diagrams to help in the representation of the movements of solid bodies. So he is confined to conceiving motion as simply change in the values of his co-ordinates as the time passes; and he finds that some kinds of change are possible, representing motions of translation and rotation, but that others are impossible, if the body is rigid, and the distances between its parts are thus to remain unaltered. The various points of the moving body are known to his Analysis only by co-ordinates or measurements of position referred to some ideal framework with respect to which it moves; and the test which is to decide whether a given mode of change of these co-ordinates represents a possible motion of the rigid body, is that the square of the distance between each two points, represented to him only as a certain quadratic function of their relative co-ordinates, is to remain constant. If the idea had lain in Lagrange's path, it would have been an easy matter to examine the character of those changes of position that do not satisfy this condition, and thus investigate how far this notion of distance is involved in an idea of space, which regards it as merely a simple *continuum*, in which movement or change of position can go on. This question has actually received its answer at a much later date from von Helmholtz.

But if in highly abstract fundamental discussions of this kind, the release from current habits of thought, which is conferred by purely analytical reasoning, is a powerful factor in Analysis and generalization, we may recall that on the other hand the first account which appeared, of the possibilities of wider laws of space-relation than the ones that belong to our experience, came from Lobatchewsky of Kasan, whose centenary has just

been celebrated, and that they proceeded on the lines of strictly Euclidean development.

In these remarks we have allowed ourselves to stray somewhat from the subject we began with, which was to survey the characteristics of the geometrical method of investigation, and take note of the qualities that belong to it. Our conclusion will perhaps be that there is no strict line of demarcation between Geometry and Analysis; that each method flourishes by means of the aid derived from the other, that when the labyrinth of Analysis becomes too complicated to be threaded by any finite clue, we must hail back to a geometrical standpoint for new fundamental conceptions; that on the other hand when the possibilities of wider geometrical domains than we are accustomed to are under investigation, we can remove our prepossessions as to space-relationship, however deep-seated, by casting the processes into a severely algebraic mould. But when all is counted up on both sides, there remains the fact that Analysis itself approaches perfection by coming closer to geometrical ideas, so that the geometrical method is the master-key in discussions about continuously varying magnitude.

And this supplies a sufficient reason why the mode of formulation of the fundamental principles of Geometry, whether for philosophical or for educational purposes, has always been considered so essential a part of the general subject of scientific method. It may also allow us to entertain the view that in geometrical instruction there are two distinct ends to be aimed at; we must not neglect the severe training in the formulation of the logical basis of knowledge which is involved in a valid presentation of the principles of the Science; nor, on the other hand, must a too strict regard for logical form prevent the acquisition of that almost intuitive familiarity with the properties of figures in space, which is derived from practice in easy problems of Pure Geometry, and forms one of the most valuable trainings for work in most branches of exact knowledge.

J. Larmor.

ANNUITIES TREATED WITHOUT PROGRESSIONS.

The method usually given in text books on Algebra for finding the present value of an annuity to last for a given number of years, or the accumulated amount of an annuity left unpaid for a given time, consists in writing down the present values or amounts of the several payments, and summing them by

the usual formula for a geometrical progression. The following method, by which the result is found from first principles without the use of formulae, is so simple and instructive that I am rather surprised that the idea does not appear to have suggested itself to teachers and writers of text books before now, or, if it has done so, that it has received so little notice.

The method was first applied by me to the solution of one of the questions in the *Intermediate Arts Directory* for July, 1893. The following proofs are taken by permission from the English revised edition of Radhakrishnan's *Text Book of Algebra*, now in course of publication under the title of *The Tutorial Algebra* (Clive & Co.), where they will appear in the chapter on Interest and Annuities, together with the ordinary treatment which affords an alternative verification of the results. It is hoped that the present treatment may prove interesting to readers of the Gazette.

The method in every case is to first find the principal which would have to be invested at the given rate of interest to produce the given annuity while it remained invested. The value of the annuity is the difference between the values of this principal at the times when the annuity begins and ends respectively.

In numerical examples the student should be encouraged in working the results from first principles and not employing formulae, and to show how this may be done examples on each proposition except the first are appended below.

(1) *To find the present value of a perpetual annuity.*

The present value of a perpetual annuity A is evidently such a sum as will bring in interest to the amount of A per annum when invested.

Hence, if r be the ratio of the interest for one year to the principal and P the present value, we have

$$Pr=A,$$

$$\therefore\ P=\frac{A}{r}=\frac{100A}{c},$$

where c is the rate *per cent.*

(2) *To find the present value of an annuity to continue for any number of years.*

Let A be the annuity, n the number of years, r the ratio of the interest, and R that of the amount for one year to the principal at the given rate of interest.

As in the last article A/r represents the sum on which the yearly interest is A.

If this sum remain invested for n years we shall obtain an annuity A for n years, and at the end of that time the original principal A/r will be repaid.

Now the present value of A/r repaid at the end of n years is

$$\frac{A}{rR^n}.$$

Hence the present value of the annuity alone is

$$P=\frac{A}{r}-\frac{A}{rR^n}=\frac{A(1-R^{-n})}{r}.$$

Example.—Find the present value of an annuity of £30, to continue for 40 years, the interest being at the rate of 6 per cent.

The sum which must be invested at 6 per cent. to bring in £30 per annum is

$$£30\times 100\div 6=£500.$$

The annuity could therefore be obtained by investing £500 and recovering the principal after 40 years.

$$\therefore \text{ present value } P=500-\frac{500}{(1{\cdot}06)^{40}}=500\{1-(1{\cdot}06)^{-40}\}.$$

By means of a table of logarithms we find that

$$\log(1{\cdot}06)^{-40}=-40\log 1{\cdot}06=-40\times{\cdot}0253059=\bar{2}{\cdot}9877640=\log{\cdot}0972219;$$

$$\therefore \text{ required present value}=£500(1-{\cdot}0972219)$$
$$=£500\times{\cdot}9027781=£451{\cdot}38905=£451\ 7s.\ 9\tfrac{1}{2}d.$$

(3) *To find the present value of a deferred perpetual annuity, of which the first payment takes place in $p+1$ years' time.*

Such an annuity A could be obtained by investing a sum of A/r in p years' time, and the present value of that sum is AR^{-p}/r, which is therefore the present value of the annuity.

(4) *To find the present value of an annuity deferred for p years and continuing for q years.*

A deferred annuity A to continue for q years, the first payment taking place at the end of $p+1$ years from the present date, could be obtained by investing a sum of A/r in p years' time, and leaving it invested for q years.

Hence the required value is the difference between the present value of A/r due p years hence, and the present value of the same sum due $p+q$ years hence, that is

$$P=\frac{A}{rR^p}-\frac{A}{rR^{p+q}}=\frac{A}{r}(R^{-p}-R^{-p-q}).$$

Example.—An estate worth £500 per annum is let on a lease of 99 years, with the option of renewal at the end of 20 years on payment of a fine. Find the fine to be paid if interest be allowed at the rate of 5 per cent.

Since the estate is worth £500 per annum at 5 per cent. interest, it is equivalent to an invested capital of

$$£500\times 100\div 5=£10{,}000.$$

When the lease has run for 20 years it will revert to the owner in 79 years, hence its (then) present value to the owner

$$=£10,000 \div (1{\cdot}05)^{79} = £10,000 \times (10{\cdot}5)^{-79}.$$

If, however, the lease is renewed it will run for 99 instead of 79 years, and its present value to the owner

$$=£10,000 \div (1{\cdot}05)^{99} = £10,000 \times (1{\cdot}05)^{-99}.$$

Hence the fine or amount to be paid for the extension of the lease is

$$£10,000 \times \{(1{\cdot}05)^{-79} - (1{\cdot}05)^{-99}\}.$$

[*Or*, we might simply say that the extension of the lease is equivalent to giving the leaseholder the use of £10,000 for the additional period of from 79 to 99 years after the date of renewal, giving the same result as before.]

By logarithmic computation we find that

$$(1{\cdot}05)^{-79} = {\cdot}0211858 \text{ and } (1{\cdot}05)^{-99} = {\cdot}0079849,$$

$\therefore$ the fine $= 10,000({\cdot}0211858 - {\cdot}0079849) = 10,000 \times {\cdot}0132009 = £132$ and a fraction.

(5) *To find the amount of an annuity left unpaid for any number of years.*

By investing A/r for n years we obtain an annuity A, and at the end of n years we recover our original principal A/r. If, however, the interest is left unpaid, then the total accumulated amount at the end of n years, reckoning compound interest, is $A/r \times R^n$. The difference represents the accumulated amount of the unpaid interest, *i.e.*, the amount of an annuity of A left unpaid for n years. Hence if M be the required amount we have

$$M = \frac{A}{r}R^n - \frac{A}{r} = \frac{A}{r}(R^n - 1).$$

Example.—Find the amount of an annuity of £200 left unpaid for 15 years, the rate of interest being $2\frac{1}{2}$ per cent.

The sum which must be invested at $2\frac{1}{2}$ per cent. to bring in £200 annually is

$$= £200 \times 100 \div 2\tfrac{1}{2} = £8000.$$

If the interest remained unpaid for 15 years, the total accumulated amount at compound interest would be

$$£8000 \times (1{\cdot}025)^{15},$$

while if interest were paid annually, only the original principal, or £8000, would be repayable at the end.

Hence the accumulated amount of the unpaid annuity

$$= £8000\{(1{\cdot}025)^{15} - 1\}.$$

Now $\quad \log(1{\cdot}025)^{15} = 15 \log 1{\cdot}025 = 15 \times {\cdot}0107239 = {\cdot}1608585 = \log 1{\cdot}4483.$

Hence the required amount

$$= £8000(1{\cdot}4483 - 1) = £(8000 \times {\cdot}4483) = £3586\ 8s.$$

(6) A similar method may also be employed to find the present value of an annuity in which the payments increase in arithmetical progression each year, a problem which, by the ordinary method involves the summation of an arithmetico-geometrical series. This is best illustrated by means of an example.

Example.—Find the present worth of a perpetual annuity of £1, payable at the end of the first year, the yearly payments increasing by £1 each year, and the rate of interest being $3\frac{1}{3}$ per cent.

Here £1 is the annual interest on £30.

To obtain £1 interest at the end of the first year we should therefore have to invest £30.

To obtain £2 interest at the end of the second year we should have to invest another £30 in one year's time, making £60 in all.

To obtain £3 interest at the end of the third year we should have to invest another £30 in two years' time, and so on.

Hence the given annuity is equivalent to a series of payments of £30 once a year, *i.e.*, to a perpetual annuity of £30, the first payment taking place immediately.

Therefore by (1) its present value $=$ £$30 \times 30 +$ £$30 =$ £930.

It will now be easy for the teacher or student to apply the same methods to almost any problem on annuities that may be proposed.

G. H. BRYAN.

THE CONIC DETERMINED BY FIVE GIVEN POINTS.

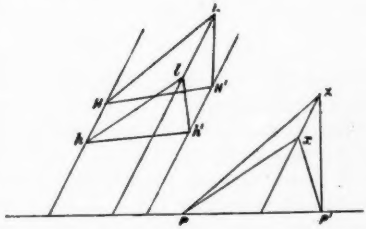
The majority of text-books on Geometrical Conics do not give any proof of the fundamental property that one and only one conic passes through any five given points, no three of which lie on a straight line. Special attention is drawn to this omission by the Rev. Dr. C. Taylor in an article published in the *Gazette* for May, 1895. The following is a description of three methods which various attempts at obtaining an elementary proof of the theorem have suggested. It is, however, very desirable to have a simpler proof than any of these, if such can be given.

We cannot expect to be able to prove the theorem in question by simply showing that there is a fixed point S such that the distance of each of the given points from S bears the same ratio to its distance from a fixed straight line. For in order to show this we must find S, which is a focus of the conic through the given points. But a construction for one focus will determine all the foci; and such a construction will involve first finding the centre and axes of the conic. Hence, in whatever way we attempt to prove the theorem, we shall, in one form or another, have to make use of the properties of the centre and diameters.

One proof of the theorem, which will only be referred to here, can be deduced from the property that if through any point T two straight lines be drawn in fixed directions, cutting a given conic in P, P' and Q, Q', then the ratios of the products $TP \,.\, TP'$, $TQ \,.\, TQ'$ is constant for all positions of T. The objection to this method of proof is the necessity of paying attention to the sign of each product.

A second proof, which, however, cannot be rendered general without appealing to the principle of continuity, is by oblique parallel projection; which, when effected in the plane, consists in altering the ordinates of points in a fixed direction to a given base line in any constant ratio. Thus parallel lines project into parallel lines; and it can be proved that any oblique projection of a circle is an ellipse.

Let H, H', L, M, N be the five given points. We may suppose that L, M, N are all on the same side of HH'. For the sake of simplicity the points M, N, and those points mentioned later whose construction depends on M, N, are not inserted in the figure. Take any point X and any base line PP'; draw XP, XP' parallel to LH, LH'; XQ, XQ' parallel to MH, MH'; and XR, XR' parallel to NH, NH'. The locus of points at which PP', QQ' subtend equal angles (or supplementary angles if PP', QQ' are in opposite directions) is, by a known theorem, a circle. Similarly there is a second circle corresponding to PP', RR'. If these circles are real, and have a real intersection, the given points will lie on an ellipse, otherwise they will lie on a hyperbola. For suppose the circles cut in a real point x on the same side of PP' as X. Project with



respect to PP' as base line, so that x may be the projection of X; and let H, H', L, M, N project into h, h', l, m, n. Then the angles PxP', hlh' are equal; for XP, XP' are parallel to LH, LH', and their projections xP, xP' are parallel to lh, lh'. And since the angles PxP', QxQ', RxR' are equal, by construction, the angles hlh', hmh', hnh' are also equal. Thus h, h', l, m, n lie on a circle, and therefore H, H', L, M, N lie on an ellipse.

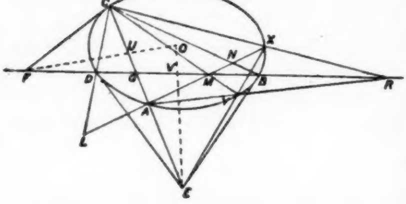
A third proof, which is free from the objections already noticed, but which assumes the properties of poles and polars, and is therefore not so elementary as might be desired, depends on the following theorem:—

If A, B, C, D be any four given points, and any line through A cut CD, DB, BC in L, M, N, then the locus of the point X on this line such that the range $LMNX$ is harmonic is a conic through A, B, C, D.

Let AC, BD cut in G; let E, F be the harmonic conjugates of G with respect to AC, BD, and U, V the middle points of AC, BD; let EV, FU cut in O. Then one and only one conic can be described with centre O, touching ED at D, and passing through B; for OD is then a semi-diameter, DE the direction of

the conjugate diameter, and B a point on the curve. This conic will pass through A and C; for the pole of BD is the point E where the diameter bisecting BD meets the tangent at D; and the polar of F is EG, since it passes through E the pole of BD, and also through G. Hence the chord of the conic on the line EG is bisected by FO at U, and is divided harmonically by E and G; and this chord can be no other than AC.

Take any point X on the conic; let EX cut the conic again in Y; let AY, CX meet in R, and AX, CY in M. Then since A, C, X, Y are four points on the conic, the triangle EMR is a self-conjugate triangle. Hence M and R lie on the polar of E, that is on BD; and the range $DMBR$ is harmonic. Therefore the harmonic pencil $C(DMBR)$ cuts AX harmonically; hence the locus of the point X such that the range $LMNX$ is harmonic is the conic we have been considering.



Let B, C, D, X, X' be any five given points. Then there is only one line through X such that the range $LMNX$ is harmonic, XM being found by joining X to the point M in which the harmonic conjugate of CX with respect to CB, CD cuts BD. Let XM meet the similar line $X'M'$ through X' in A. Then B, C, D, X, X' lie on a conic, viz., the conic corresponding to A and the triangle BCD as in the theorem.

EDITOR.

EXAMINATION QUESTIONS AND PROBLEMS.

Questions 62-69 *are taken from Woolwich and Sandhurst Entrance papers set in November last;* 70-81 *from recent Entrance Scholarship papers at Oxford and Cambridge; and* 82, 83 *from the London Matriculation paper of January.*

62. Simplify

$$\frac{1}{1-x^4}\left\{1+\frac{x}{1-x}+\frac{x^2}{1-x^2}+\frac{x^3}{(1-x)(1-x^2)}+\frac{x^3}{(1-x^2)(1-x^3)}\right\}.$$

63. Show that a rectangle of given perimeter has the greatest area when its sides are equal.

64. From a given point without a circle draw, when possible, a straight line which shall have its middle point and its other end upon the circumference.

65. The sum of the squares of the distances of a fixed point on a circle from the ends of a variable chord is constant; prove that the locus of the middle point of the chord is a straight line.

66. Prove that $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}=\sin^{-1}\frac{63}{65}$.

67. Show how to solve a triangle when the lengths of the medians are given.

68. From a ship sailing in a north-easterly direction at 15 miles an hour it is observed that a second ship is always due south; supposing that this second ship is sailing eastwards find the rate at which it is travelling.

69. A ladder AB rests at an inclination of $\tan^{-1}\frac{1}{2}$ against the ground at A and a vertical wall at B, the coefficients of friction at A and B being $\frac{2}{3}$ and $\frac{1}{2}$. If the centre of gravity of the ladder is 6 feet from A and 9 feet from B, find how far a boy, whose weight is $\frac{1}{5}$ that of the ladder, can ascend before it begins to slip.

70. Solve the equations

$$x^2+y^2=z^2-12, \quad zx+y=13, \quad yz-x=13.$$

71. Out of a pack of 52 cards two have been lost. Three cards are drawn from the remaining 50 and found to be all aces; find the chance that a fourth drawing may also give an ace.

72. Through the vertex A of a square $ABCD$ a straight line is drawn meeting CB, CD, DB at E, F, K respectively; prove that CK touches the circle through C, E, F.

73. A hoop in the form of a parabola is held fixed in a vertical plane with its axis horizontal. A perfectly elastic particle is dropped from the upper end of the latus rectum; find the time and place of its second impact with the hoop.

74. The equation $3x^4-8x^3+3x^2-2x-2=0$ is known to have two roots whose product is -1; find all the roots.

75. If PQ is a diameter of a rectangular hyperbola, and PR a chord normal to the curve at P, prove that PQR is a right angle.

76. A hyperbola cuts a concentric ellipse orthogonally at four points, not being a confocal. Prove that the tangents to the ellipse at adjacent points of intersection are perpendicular to one another.

77. Prove that, if $\dfrac{b-c}{y-z}+\dfrac{c-a}{z-x}+\dfrac{a-b}{x-y}=0$,

then $(b-c)(y-z)^2+(c-a)(z-x)^2+(a-b)(x-y)^2=0$.

78. A chord PQ of a conic passes through a fixed point. If the circle on PQ as diameter meets the conic again in P', Q', show that $P'Q'$ also passes through a fixed point.

79. Solve the equations

$$u(2a-x)=x(2a-y)=y(2a-z)=z(2a-u)=b^2;$$

and prove that unless $b^2=2a^2$, $x=y=z=u$; but that if $b^2=2a^2$, the equations are not independent.

80. Prove that the expression $\dfrac{x+a}{x^2+bx+c}$ will always lie between two fixed finite limits if $a^2+c^2>ab$, and $b^2<4c^2$; that there will be two limits between which it cannot lie if $a^2+c^2>ab$, and $b^2>4c^2$; and that it will be capable of all values if $a^2+c^2<ab$.

81. If $x^3=234\sqrt{2}+148\sqrt{5}$, and x is real, prove that

$$x+\frac{2}{x}=4\sqrt{5}.$$

82. Prove that $(a-bc)(b-ca)(c-ab)+(1-a^2)(1-b^2)(1-c^2)$ is divisible by $1-abc$, and find the quotient.

83. If a, b, c are in A.P., and x, y, z in G.P., prove that

$$x^b y^c z^a = x^c y^a z^b.$$

84. Point out the fallacy in the following: If x, y be equal, and not zero, then $x=y=\dfrac{x^2}{x}=\dfrac{y^2}{y}=\dfrac{x^2-y^2}{x-y}=x+y$, which is absurd. G. H. WARD.

85. If $a-b$, $a-c$, $a-d$ are in H.P., then so also are $b-c$, $b-d$, $b-a$, and $c-d$, $c-a$, $c-b$, and $d-a$, $d-b$, $d-c$. PROF. STEGGALL.

86. What is the locus represented by the equation

$$\{y^2+(a-x)^2\}\{x^2+(b-y)^2\}=\{ax+by-(x^2+y^2)\}^2.$$

Account geometrically for the result. E. M. LANGLEY.

87. If a tangent be drawn to a parabola at Q, and from any point O on the tangent a line OPD be drawn parallel to the axis meeting the curve in P, and QD be drawn perpendicular to OP, prove that $4AS\,.\,OP=QD^2$. T. ROACH.

88. Prove that the difference of the cubes of two consecutive integers is a number terminating in one of the digits 1, 7, 9. Is it possible for this difference to be a perfect square? R. F. DAVIS.

89. If c and d are the positive and negative roots of the equation $(x+1)(x-\sqrt{5})=1$, prove that

$$(c+d\sqrt{3})\operatorname{cosec}1^\circ-(c\sqrt{3}-d)\sec 1^\circ=8\sqrt{2}.$$ EDITOR.

90. In a spherical quadrilateral the arcs joining the middle points of opposite sides and the arc joining the middle points of the diagonals are concurrent. PROF. NEUBERG.

(A correspondent requests a trigonometrical solution of this problem, which is taken from Casey's *Spherical Trigonometry*.)

SOLUTIONS.

18. *Soient OA, OB, OC trois droites situées dans un même plan. Par un point M de OC on mène une transversale quelconque qui rencontre OA en E, OB en F, et l'on joint les points E, F à un second point fixe N de OC. Démontrez que la différence $\cot ENO - \cot FNO$ est constante.*

PROF. J. NEUBERG.

The solution depends upon the following theorem which is very useful in Statics: If P be any point in the base BC of a triangle ABC,

$$BC \cot APC = PC \cot B - BP \cot C.$$

This may be proved by drawing AN perpendicular to the base; then

$$BC \,.\, PN = BN \,.\, PC - BP \,.\, NC;$$

hence, dividing by AN, the theorem mentioned follows.

Applying the theorem to the question proposed,

$$ON \cot EMN = MN \cot AOC - OM \cot ENO,$$

and also $\quad ON \cot EMN = MN \cot BOC - OM \cot FNO;$

hence $\quad OM(\cot ENO - \cot FNO) = MN(\cot AOC - \cot BOC)$; etc.

20. *On donne un triangle ABC. Par les sommets B, C on mène deux parallèles qui rencontrent les côtés opposés AC, AB aux points E, F. Démontrez que la droite EF est tangent à une hyperbole fixe lorsque la direction des parallèles change.*

PROF. J. NEUBERG.

The triangles AEF, ABC are equal in area; and EF envelopes a hyperbola having AB, AC for asymptotes and BC for a tangent.

22. *If R is the resultant of forces P and Q, S of forces P and R, and T of forces Q and R, prove that $S^2 + T^2 = P^2 + Q^2 + 4R^2$.*

W. GALLATLY, M.A.

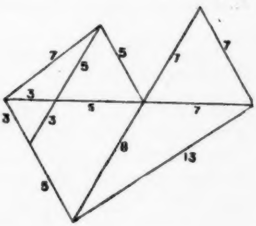
Completing the parallelograms, it is easily seen that

$$\left.\begin{aligned} S^2 + Q^2 &= 2P^2 + 2R^2 \\ T^2 + P^2 &= 2Q^2 + 2R^2 \end{aligned}\right\}.$$

26. *Illustrate by a single diagram that triangles whose sides are proportional to any one of the following sets of numbers have either an exterior or an interior angle of 60°:* (3, 5, 7), (5, 7, 8), (7, 8, 13), (7, 13, 15), (8, 13, 15).

E. M. LANGLEY.

The figure required is as follows:



There is a sixth triangle fulfilling the condition: (3, 8, 7). PROF. W. H. HUDSON.

30. *Defining the parabola as the path of a projectile under a force constant in direction and magnitude, show that the locus of the mid-points of a set of parallel chords is a straight line.*

H. D. DRURY, M.A.

If two particles describe the same curve under the same forces but in opposite senses it is *assumed* that their velocities at any point of the curve are equal and opposite.

Let, therefore, two particles acted upon by gravity only, be simultaneously projected from P, one in the direction PT, and one in the opposite direction PT', with equal velocities u; then, by the assumption, they describe portions of the same parabola. To find the points Q, Q' on the curve where they will be at the end of τ seconds after projection, make $PT = PT' = u\tau$ feet, and draw TQ, TQ vertically downwards each $= \frac{1}{2}g\tau^2$ feet. It is thus seen that QQ' is always parallel to TT', and that the middle point of QQ' lies on the vertical line through P.

36. *A plane cuts OZ, the axis of a right circular cylinder, at right angles in O; BOB′, DOD′ are two diameters of the cylinder at right angles to each other in the given plane. The given plane revolves round BOB′. Find the locus of either of the foci of the variable ellipse thus generated; and show that DT, a straight line through D parallel to the axis of the cylinder, is an asymptote to the curve.* LIEUT.-COL. H. W. L. HIME.

In the plane containing the axis OZ, the diameter DOD' and the generating lines DT, $D'T'$, draw any straight line AOA' through O, meeting DT, $D'T'$ in A, A' respectively. Then AOA', BOB' are the axes of the section of the cylinder made by a plane through AOA' and BOB'. If upon AA' points S, S' be taken, such that

$$AS \, . \, SA' = AS' \, . \, S'A' = OB^2 = a^2,$$

then S, S' are the two foci.

If r, θ be the polar coordinates of S, we have

$$AS \, . \, SA' = a^2 \sec^2\theta - r^2 = a^2, \qquad \therefore \; r = \pm a \tan\theta.$$

When $\theta = \frac{\pi}{2}$, $r = \infty$; and at the same time SA vanishes, so that DA or DT is an asymptote.

38. *Find by an elementary method the maximum value of*

$$\cos\theta\{\sin\theta + \sqrt{(\sin^2\theta + \sin^2\alpha)}\}.$$ E. M. LANGLEY.

Put $$u = \cos\theta\{\sin\theta + \sqrt{(\sin^2\theta + \sin^2\alpha)}\};$$
then $$(u - \sin\theta \, . \, \cos\theta)^2 = \cos^2\theta(\sin^2\theta + \sin^2\alpha);$$
$$u^2 - 2u\sin\theta \, . \, \cos\theta = \sin^2\alpha \, . \, \cos^2\theta;$$
$$u^2\tan^2\theta - 2u\tan\theta + u^2 - \sin^2\alpha = 0.$$

Hence, in order that $\tan\theta$ may be real, we have

$$u^2 \gtreqless u^2(u^2 - \sin^2\alpha);$$

and dividing out by the positive quantity u^2,

$$u^2 \lesseqgtr 1 + \sin^2\alpha.$$

Hence u lies between $\pm\sqrt{(1+\sin^2\alpha}$; and the maximum value of u is therefore $\sqrt{(1+\sin^2\alpha)}$.

40. *If Y be the foot of the perpendicular from S on the tangent at the point P of an ellipse, the angle PSC is double of the angle SYA.* P. J. HEAWOOD.

Let $S'Y'$ be the perpendicular from the focus S' to the tangent at P. Then we know that Y, Y' lie on the circle on AA' as diameter, and that CY' is parallel to SP.

The angles AYA', SYY' are right angles; and therefore SYA is equal to $A'YY'$; but $A'YY'$ is half the angle at the centre $A'CY'$ which stands on the same arc $A'Y'$. Hence SYA is half of $A'CY'$, or half of PSC.

43. *A bullet is fired and hits a wooden target horizontally, and penetrates to a depth of* 1 *foot. When the target is* 9 *inches thick the bullet fired under the same conditions afterwards strikes the horizontal ground* 400 *yards farther on. How far from the firing point is the target?* J. H. HOOKER.

Assuming that the horizontal pressure between the bullet and the target is constant so long as the bullet has not been brought to rest, the work done against the bullet is proportional to the distance it has penetrated. In penetrating 12 inches the kinetic energy of the bullet becomes completely destroyed; hence after penetrating 9 inches three-fourths of the kinetic energy has been lost, and the bullet still retains one-fourth, *i e.*, its velocity is half that which it had on arriving at the target. This velocity carries the bullet 400 yards horizontally in the time it falls to the ground, and if no

velocity had been destroyed by the target the distance would have been twice as great, or 800 yards. The firing point, if on the ground, is therefore 800 yards distant horizontally from the target; but, as the height of the target is not given, the actual distance of the firing point from the target is not known.

44. *BE, CF are perpendiculars from B, C to the opposite sides CA, AB of the triangle ABC; EG, FH are perpendiculars on CF, BE respectively. Prove that EF is a mean proportional between BC and GH.* R. TUCKER.

The points B, F, E, C lie on a circle, and the triangles ABC, AEF are therefore similar. Hence

$$EF/BC = EA/AB = \cos A.$$

Similarly, if EG, FH meet in D,

$$GH/EF = \cos D = \cos A;$$

$$\therefore \quad EF/BC = GH/EF, \text{ or } EF^2 = BC \cdot GH.$$

45. *ABC is any triangle; S any point. Show that it is possible to construct an equilateral triangle DEF such that SD, SE, SF are equal to BC, CA, AB respectively. Show also that the square on one of its sides is*

$$\tfrac{1}{2}(a^2+b^2+c^2)+2\sqrt{3}\Delta.$$ T. H. DE LAND.

Let P, Q, R be the circumcentres of the equilateral triangles described on BC, CA, AB on the sides remote from A, B, C respectively. It is easily proved that the circumcircles meet in one and the same point; let this point be O; then OP, OQ, OR are the radii of the circles, and are proportional to BC, CA, AB.

In the triangle AQR, $AQ=\dfrac{b}{\sqrt{3}}$, $AR=\dfrac{c}{\sqrt{3}}$, $Q\hat{A}R=A+60°$;

$$\therefore \quad QR^2 = \tfrac{1}{3}(b^2+c^2-2bc\cos\overline{A+60°})$$
$$= \tfrac{1}{3}(b^2+c^2-bc\cos A+\sqrt{3}bc\sin A)$$
$$= \tfrac{1}{6}(a^2+b^2+c^2+4\sqrt{3}\Delta).$$

Hence it follows that the triangle PQR is equilateral. Also (since $OP=\dfrac{a}{\sqrt{3}}$, etc.) if we produce OP, OQ, OR to D, E, F so that $OD=\sqrt{3}OP$, etc., we shall obtain an equilateral triangle DEF such that OD, OE, OF are equal to BC, CA, AB; and $EF^2=3QR^2=\tfrac{1}{2}(a^2+b^2+c^2+4\sqrt{3}\Delta)$.

46. *Démontrez que l'équation*

$$\sqrt{x-a}+\sqrt{x-b}+\sqrt{x-c}=0,$$

a toujours deux racines réelles et inégales. PROF. J. NEUBERG.

Cleared of radicals the equation may be written in the form

$$\{3x-(a+b+c)\}^2 = 2\{(b-c)^2+\ldots+\ldots\};$$

and since the right-hand side is necessarily positive, the roots of the equation are real.

47. *Pour quelles valeurs de m les racines de l'équation*

$$x^2-4(1-m)x+(1-m^3)=0$$

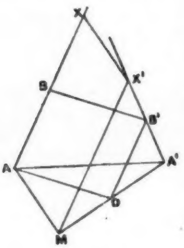
sont-elles réelles, inégales, ou imaginaires? PROF. NEUBERG.

From the given equation we have

$$\{x-2(1-m)\}^2 = -(1-m^3)+4(1-m)^2$$
$$=(m-1)(m^2+5m-3)$$
$$=(m-1)\left(m-\frac{\sqrt{37}-5}{2}\right)\left(m+\frac{\sqrt{37}+5}{2}\right).$$

Hence, when $m=1$ or $\frac{\sqrt{37}-5}{2}$ or $\frac{-\sqrt{37}-5}{2}$, the roots of the equation are equal. When m lies between $+\infty$ and 1, the factors on the left hand are all positive, and the roots are real; when m lies between 1 and $\frac{\sqrt{37}-5}{2}$, the first factor is negative and the other two positive, and the roots are imaginary; similarly, when m lies between $\frac{\sqrt{37}-5}{2}$ and $\frac{-\sqrt{37}-5}{2}$, the roots are real, and when m lies between $\frac{-\sqrt{37}-5}{2}$ and $-\infty$, the roots are imaginary.

48. *Deux droites AB, $A'B'$, non situées dans un même plan, sont coupées par un plan P aux points X and X'. Lorsque le plan P se meut parallèlement à lui-même, qui est le minimum de la droite XX'?* PROF. NEUBERG.



Let AA', BB' be any two fixed positions of the line XX'; draw AD equal and parallel to BB', and AM equal and parallel to any line XX'. Then AA', AD, AM are all parallel to the plane P, and therefore lie in a fixed plane; also $X'M$ and $B'D$ are both parallel to XA, and lie in the plane through $A'B'$ parallel to AB. Hence A', D, M lie on a straight line which is the intersection of the two planes above-mentioned. The line XX' will be a minimum when AM is a minimum, *i.e.*, when AM is perpendicular to $A'D$.

If AA' and BB' are equal, the point M bisects $A'D$; and the minimum line XX' bisects AB and $A'B'$, and is equally inclined to AA' and BB'.

49. *If* $\frac{1}{6841t}$ *(scale* 11*) be converted into an infinite undecimal, prove that the* 49513*th figure is* 5. W. P. WORKMAN.

If p is a prime, and prime to r, then by Fermat's theorem p is a factor of $r^{p-1}-1$. Hence $1\div p$, when converted to a decimal in scale r, will repeat after $p-1$ figures. Thus the number of figures in the period of $1\div p$ is $p-1$ or a factor of $p-1$; and if this number is even, say $2n$, it can be proved that any figure and the nth thereafter together make $r-1$. Now $6841t$, when expressed in the decimal scale is 98999. This is a prime. Also $98998=2\times 49499$, and the latter factor is also a prime. Hence by the properties of circulating decimals, the period due to the fraction $1\div 6841t$ must contain either 49499 figures or 98998; and it may be supposed we do not know which. If the former, then the 49513th is the same as the 14th. If the latter, then the 14th and the 49513th must together be equal to 10. Now by actual division the first 14 figures in the decimal are $\cdot 000016992991t5$. Therefore, whichever be the period, the 49513th figure must be 5.

GEORGE HEPPEL, M.A.

50. *The sum of the digits of every multiple of* 2739726 *up to the* 72*nd is* 36. E. M. LANGLEY.

Since $2739726=9999\times 274$, it will be sufficient to prove that every even multiple of 9999 up to the 20000th has the sum of its digits equal to 36.

(i.) Consider any multiple of 9999 up to the 10000th. If $x<10000$, the number $9999(x+1)$, or $10^4x+(9999-x)$, has not more than 8 digits; and if a, b, c, d are the digits of x, those of $9999(x+1)$ are

$$a,\ b,\ c,\ d,\ 9-a,\ 9-b,\ 9-c,\ 9-d,$$

so that their sum is 36.

(ii.) Consider any even multiple of 9999 from the 10002th to the 20000th. Then since $9999 \times 10001 = 10^8 - 1$; if we add to this any odd multiple of 9999 up to the 9999th, *i.e.*, if we add a number whose digits are

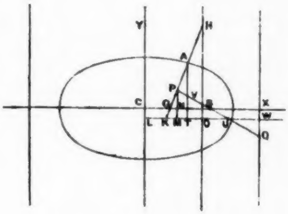
$$a,\ b,\ c,\ d,\ 9-a,\ 9-b,\ 9-c,\ 9-d,$$

(the last being odd), we get a number whose digits are

$$1,\ a,\ b,\ c,\ d,\ 9-a,\ 9-b,\ 9-c,\ 8-d;$$

and the sum of these is 36. GEORGE HEPPEL, M.A.

52. *Four normals are drawn from a given point P to an ellipse meeting it in A, B, C, D. The rectangular hyperbola passing through these five points cuts the directrices of the ellipse in Q and R. Prove that the straight lines PQ, PR pass through the foci of the ellipse.* E. P. ROUSE.



Let A be any point (not a vertex) of an ellipse centre C, axes CX, CY. Take any point P on the normal AG; draw PN perpendicular to CX, and produce it to M, so that $PN = e^2PM$; draw ML perpendicular to CY, and produce LM to O, so that $LM = e^2LO$; draw OH parallel to CY.

Then A lies on the rectangular hyperbola having OL, OH for asymptotes which passes through P. For

$$e^2KM = GN = CN - CG = e^2(LO - LT);$$

$\therefore\ KM = TO$ and $KP = AH$.

The hyperbola intersects the ellipse in *four* points, the normals at which all pass through P. If the hyperbola intersects the S-directrix in Q and PQ meets OL, OH in U, V, then $PU = VQ$, or $MU = OW$. Therefore

$$e^2MU = e^2OW = e^2(CX - LO) = CS - CN = NS;$$

therefore PQ passes through S.

55. *Show that* $\dfrac{a+1}{a-1}$ *is an approximate value of* $\sqrt{\dfrac{a+2}{a-2}}$, *the approximation being closer as a increases. Hence show that, approximately,*

$$\sqrt{13} = 18/5,\quad \sqrt{77} = 351/40,\quad \sqrt{2} = 99/70,\quad \sqrt{47} = 665/97.$$

A. M. WILCOX.

Since $\dfrac{a+2}{a-2} - \left(\dfrac{a+1}{a-1}\right)^2 = \dfrac{4}{(a-2)(a-1)^2}$, the error diminishes constantly as a increases, while at the same time the correct value of the square root approaches unity as its limit.

Or thus:

$$\sqrt{\frac{a+2}{a-2}} - \frac{a+1}{a-1} = \left(1+\frac{2}{a}\right)\left(1-\frac{4}{a^2}\right)^{-\frac{1}{2}} - \left(1+\frac{1}{a}\right)\left(1-\frac{1}{a}\right)^{-1}$$

$$= \left(1+\frac{2}{a}\right)\left(1+\frac{2}{a^2}+\dots\right) - \left(1+\frac{1}{a}\right)\left(1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\dots\right)$$

$$= \frac{2}{a^3}+\dots.$$

The approximations stated are found by putting $a = 11, 79, 34, 96$.

REVIEWS AND NOTICES.

Mensuration for Senior Students. By ALFRED LODGE, M.A. (Longmans.) An excellent treatise, profusely illustrated with well-chosen and well-executed diagrams, and intended for students with a fair knowledge of Algebra and Trigonometry. The student is shown how known geometrical theorems bear on his formulæ, and how to adapt the latter to the practical needs of the arithmetical computer.

Elementary Mensuration. By F. H. STEVENS, M.A. (Macmillan & Co.) A serviceable treatise, formed of two concurrent courses. The first provides for those whose knowledge of Geometry is confined to Euclid's Book I., and of Algebra to the meaning of the simplest symbols. The second supplies matter for those who have mastered Proportion, Algebra to Binomial Theorem, and the elements of Trigonometry. We are glad to see some attention given to contracted methods in arithmetical work, and that the application of Trigonometrical Tables to the evaluation of the formulæ of Mensuration is explained and illustrated. It is encouraging, too, to note that in the logarithmic calculations Mr. Stevens makes, like Professor Lodge, no use of "big L," but uses the actual logarithms of the functions which occur in the formula. The arrangement of the type and the execution of the diagrams seem very good.

Mechanics—Hydrostatics. By R. T. GLAZEBROOK, M.A., F.R.S. (Cambridge University Press.) A companion volume to that on Dynamics by the same author reviewed in No. 5. It seems to have the same merits and to deserve the same recommendation as its predecessor.

First Stage Mechanics. By F. ROSENBERG, M.A. (W. B. Clive.) This treatise seems carefully drawn up, and likely to be very useful to the students for whom it is intended—those of the Elementary Stage of the S. and A. Department in the Theoretical Mechanics of Solids.

Elementary Mechanics. By W. BRIGGS, M.A., and G. H. BRYAN, F.R.S. (W. B. Clive.) We have called attention in a previous number to the excellent Text-books on Dynamics and Statics by the same authors. This is a more elementary work, but seems in common with *Elementary Hydrostatics*, by the same authors, to have the same valuable qualities as its predecessors.

Mechanics for Beginners. By W. GALLATLY, M.A. (Macmillan & Co.) This book explains the first principles of Statics as far as the description of machines, which are well described as transformers of work from an inconvenient to a convenient form, and those of Dynamics, including potential and kinetic energy. In the Statics it is stated that four items are necessary for the complete specification of a force, the single item of direction being unnecessarily divided into two, viz. line of action and sense. It should not be assumed, at any rate in a book for beginners, that the theorem of moments for parallel forces follows from the like theorem for forces which are not parallel. In the Dynamics uniform acceleration is very clearly explained, and the formulæ in connection with it well worked out; but impact and normal acceleration are not so satisfactorily treated.

(i.) **Plane and Solid Geometry.** By W. W. Beman and D. E. Smith. (Ginn & Co.)

(ii.) **Elements of Geometry**. By G. C. Edwards. (Macmillan & Co.)

These are American treatises, Messrs. Beman and Smith being Professors in Michigan, and Mr. Edwards Associate Professor in the University of California. They give us a high opinion of the zeal with which American teachers try to improve the study of geometry. Both afford evidence of thought and care, and show that the authors have kept themselves abreast with European improvements. The influence of Henrici and other writers on Modern Geometry is plainly visible. Messrs. Beman and Smith's work is particularly interesting to members of the A.I.G.T., as the sequence of propositions is avowedly in the main that of our *Syllabus* and *Text-Book.* Both deserve high praise for the numerous and beautifully executed diagrams, which have a vigour and suggestiveness about them which we could wish to be more general. Both deal with Solid and Spherical as well as with Plane Geometry, Mr. Edwards adding a useful chapter on Conic Sections. Both contain excellent remarks on the solution of Exercises. The value of the first treatise is much enhanced by the frequent insertion of historical notes, by a biographical table, and by an index of the mathematical terms used. The only feature which we do not like is the somewhat inadequate treatment of proportion. Mr. Edwards boldly gets rid of the difficulty by stating that he thinks that the proper place for the theory of proportion is in the Algebra, and that for that reason he has omitted it. The authors of the other text-book supply a short algebraic treatise. But whatever may be said in favour of allowing a pupil to proceed with the study of similar figures, without waiting until he has thoroughly mastered the difficulties of Euclid's treatment, we are inclined to think no treatise on geometrical proportion complete which does not give a thorough discussion of the theory of proportion based on *quantuplicity.* The cumbrous nomenclature of Book V. might be dropped, and a convenient notation adopted, as suggested by De Morgan, and carried out in the A.I.G.T. *Text-Book of Geometry.*

Pitt Press Euclid, XI., XII. By H. M. Taylor. This is the final volume of a work marked throughout by vigour and originality, which will be welcomed by all who wish to have an improved text which does not depart from the order of Euclid's Elements. We congratulate Mr. Taylor on the completion of his edition, and take this opportunity of expressing our regret for the physical difficulties under which he has had to labour. Among much interesting matter the volume contains short expositions of Perspective, Spherical Geometry, Rotation, and Reflexion. A general index to the complete work is added.

Practical Plane and Solid Geometry. By Harrison and Baxandall. (Macmillan & Co.) This little book, which we understand from its title-page to be drawn up to meet the requirements of the Syllabus of the Elementary Stage of the S. and A. Department, has three sections. The first (18 pp.) gives solutions of the principal problems of Elementary Plane Geometry. The second, of about the same length, deals with Graphical Arithmetic. The third, which is three times the length of the other two, is devoted to Descriptive Geometry. A set of three hundred exercises, and a list of definitions and theorems of Solid Geometry is added. Pracitcal teachers, whose opinion we have consulted, speak very highly of the merits of this work.

Elementary Algebra. By J. W. WELSFORD, M.A., and C. H. P. MAYO, M.A. (Longmans, Green & Co.) This is a very excellent treatise, well adapted for the use of thoughtful teachers and thoughtful students. Its scope is rather more extensive than its name would imply, for it includes the Binomial and Exponential Theorems, and the theory and practice of Logarithms. The explanations of rules and principles seem very carefully done, and the practical directions for manipulative work are excellent. The Remainder Theorem and the use of Detached Coefficients come in for due attention, and abundant collections of exercises are supplied to illustrate and enforce the rules. We can cordially recommend it.

Should the Metric Weights and Measures be made compulsory? is the title of a pamphlet by J. EMERSON DAWSON, Chairman of the Executive Committee of the New Decimal Association. A Select Committee of the House of Commons on Weights and Measures reported in July of last year in favour of the Metric System being at once legalized; and, after a lapse of two years, becoming compulsory by Act of Parliament. The pamphlet points out the advantages to be derived from these recommendations; and advocates the plan which is adopted on the Continent, of teaching decimals in schools before vulgar fractions. There is no doubt that this would be in some respects the simpler as well as the more useful practice, since the system of numeration in decimals is a direct outcome of that in whole numbers, while vulgar fractions, when treated as ratios, involve entirely new and more difficult conceptions. Copies of the pamphlet may be had for one penny each from the Secretary of the New Decimal Association, Botolph House, Eastcheap, London, E.C.

The Science Absolute of Space. A translation from the Latin of John Bolyai, by DR. G. B. HALSTED. (The Neomon, Austin, Texas, U.S.A., 4th edition.) To the translation is prefixed an introduction of 30 pages, dealing with the principles of Non-Euclidian Geometry, and giving an interesting account of the Bolyais, father and son. An account of some of J. Bolyai's theorems, and a comparison with Lobatschewsky's *Theory of Parallels* will be given in the next number of the Gazette.

BOOKS AND MAGAZINES RECEIVED.

Primer of the History of Mathematics. By W. W. ROUSE BALL. (Macmillan & Co.) A fascinating little volume, which should be in the hands of all who do not possess the more elaborate *History of Mathematics* by the same author.

Inland Educator Monographs, No. 2. *High School Mathematical Text-Books.* By R. S. ALEY.

The American Mathematical Monthly. September, 1895, to January, 1896.

Journal de Mathématiques Elementaires. September, 1895, to March, 1896.

Periodico de Matematica. September, 1895, to February, 1896.

Tōkyō Sūgaku-Butsurigaku Kwai Kiji, Maki no VII., Dai 1-4.

An article on *Wingate's Arithmetic*, by J. H. HOOKER, and notes by PROFESSOR STEGGALL, G. HEPPEL, R. F. DAVIS, T. G. VYVYAN, and Anon., are held over, for want of space, till the next number.

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